

Convergence of the Neumann series for the Schrödinger equation and general Volterra equations in Banach spaces

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Abstract.

The objective of the article is to treat the Schrödinger equation in parallel with a standard treatment of the heat equation. In the mathematics literature, the heat equation initial value problem is converted into a Volterra integral equation of the second kind, and then the Picard algorithm is used to find the exact solution of the integral equation. The Poisson Integral theorem shows that the Poisson integral formula with the Schrödinger kernel holds in the Abel summable sense. Furthermore, the Source Integral theorem provides the solution of the initial value problem for the nonhomogeneous Schrödinger equation. Folland's proof of the Generalized Young's inequality is used as a model for the proof of the L^p lemma. Basically the Generalized Young's theorem is in a more general form where the functions take values in an arbitrary Banach space. The L^1 , L^p and the L^∞ lemmas are inductively applied to the proofs of their respective Volterra theorems in order to prove that the Neumann series converge with respect to the topology $L^p(I; \mathcal{B})$, where I is a finite time interval, \mathcal{B} is an arbitrary Banach space, and $1 \leq p \leq \infty$. The Picard method of successive approximations is to be used to construct an approximate solution which should approach the exact solution as $n \rightarrow \infty$. To prove convergence, Volterra kernels are introduced in arbitrary Banach spaces. The Volterra theorems are proved and applied in order to show that the Neumann series for the Hilbert-Schmidt kernel and the unitary kernel converge to the exact Green function.

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1. Introduction

This article is based on a portion of my master thesis [1]. The central problems and theorems of the thesis are summarized in this article. The books of the Rubinsteins [2] and Kress [3] show how the heat equation is converted to a Volterra integral equation, which is then solved by the Picard algorithm. In this article we shall show that the Schrödinger equation has similar properties and results as the heat equation such as the Poisson Integral Theorem and the Source Integral Theorem. The similarities between the Schrödinger equation and the heat equation were used to create a theoretical framework which will give the solution to the Schrödinger problem. As much as possible, we use the books [2, 3] as guides to treat the quantum problem like a heat problem. However, the parallel between the heat equation and the Schrödinger is found to be a limited one, and we use the potential theory formalism that Kress laid down in his book in order to study the existence, and uniqueness of the solution of the Schrödinger equation. The differences between the heat operator and the quantum operator require different proofs for the uniqueness theorems, the Source Integral Theorem and the Poisson Integral Theorem. For example, the Poisson integral formula with the Schrödinger kernel is shown to hold in the Abel summable sense. In the harmonic potential theory, the problem is similarly reduced to an integral equation, but the integral equation is not of Volterra form and therefore the Neumann series does not converge automatically. One can put in a numerical parameter β so that the series converges when $|\beta| < 1$, but the value one needs for the PDE problem has $|\beta| = 1$. In the elliptic PDE problem such as Laplace's equation, Helmholtz equation, or Poisson equation, to prove convergence it is not enough to have the Banach-space operator having finite norm; the norm must be less than unity. The norm must be less than unity, because one needs to bound the norm by the geometric series, not the exponential series. Therefore, the Neumann series is not used to prove existence of a solution; instead, the Fredholm theory is used to prove existence more abstractly.

The Source Integral Theorem is formulated in terms of the source integral term and the initial integral term. The Poisson and the Source Integral theorems for the Schrödinger equation are introduced in section 2. The initial value problem can be expressed as a Volterra integral equation of the second kind with respect to time. Our main task is to use the method of successive approximation in order to prove that there exists a unique solution to the integral equation. In section 3, the article focuses on linear integral operators in arbitrary Banach spaces. In section 4, the article introduces the Volterra kernels and applies the Neumann series to give an approximation to the exact solution. Folland's proof of the Generalized Young's inequality is used as a model for the proof of lemma 8 [9]. In section 4, we introduce lemma 6 and lemma 7 in order to prove the L^∞ Volterra theorem and the L^1 Volterra theorem, respectively. Then in each Volterra theorem, we inductively apply the appropriate L^p lemma to prove that the Volterra integral operator can be solved by successive approximations. In particular, we work with Volterra integral operators \hat{Q} that go from $L^p(I; \mathcal{B})$ to itself, where $1 \leq p \leq \infty$.

These Volterra integral operators \hat{Q} are assumed to have uniformly bounded kernels such that $A : \mathcal{B} \rightarrow \mathcal{B}$. Furthermore, we only consider kernels $A(t, \tau)$ which are Volterra kernels in time. Then the Volterra theorem proves that Volterra integral equation with a uniform bounded kernel can be solved by successive approximations with respect to the topology $L^\infty(I; \mathcal{B})$. The L^p Volterra theorem proves the more general case when $L^p(I; \mathcal{B})$, and where $1 < p < \infty$. In section 5, the article covers two specific kernels, the Schrödinger kernel and the Hilbert-Schmidt kernel. In the Schrödinger case, the perturbation expansion series contains a unitary operator and a uniformly bounded potential, and we prove that the Neumann series converges.

2. The Poisson Integral and Source Integral Theorems

The wavefunction $\Psi(x, t)$ of a nonrelativistic particle in \mathbb{R}^n is a solution to the Schrödinger equation

$$H\Psi(x, t) = i\hbar\partial_t\Psi(x, t) \quad (1)$$

where H is the Hamiltonian, given by

$$H = H_0 + V = \frac{1}{2m}p^2 + V(x, t) = -\frac{\hbar^2}{2m}\Delta_x + V(x, t). \quad (2)$$

The kinetic operator $T = \frac{1}{2m}p^2$, is also known as the free Hamiltonian H_0 in nonrelativistic quantum mechanics. The complex-valued function $\Psi(x, t)$ is the wavefunction, and $|\Psi(x, t)|^2$ represents a particle density function. First, we will consider the case when there is no potential, i.e, $V(x, t) = 0$. Therefore, the free Schrödinger equation becomes

$$i\hbar\partial_t\Psi(x, t) = -a^2\Delta_x\Psi(x, t) \quad (3)$$

where

$$a^2 = \frac{\hbar^2}{2m}. \quad (4)$$

The general abstract Schrödinger equation can be realized in several concrete situations such as boundaries, weak potential (where perturbation works), slowly varying potentials (where WKB theory works), and this article is a start on a program dealing with those cases.

The Schrödinger operator and its complex conjugate can be rewritten as

$$L = a^2\Delta_x + i\hbar\partial_t \quad (5)$$

and

$$L^* = a^2\Delta_x - i\hbar\partial_t. \quad (6)$$

Then the adjoint operator of L is

$$L^\dagger = L = a^2\Delta_x + i\hbar\partial_t \quad (7)$$

with respect to the usual L^2 inner product and the imposed homogeneous boundary conditions. In other words the Schrödinger operator L is formally self-adjoint. An

important difference to notice is that the heat operator is not formally self-adjoint because the time derivative term changes sign.

The nonhomogeneous Schrödinger equation with nonhomogeneous initial data is

$$\begin{aligned} Lu(x, t) &= a^2 \Delta_x u(x, t) + i\hbar \partial_t u(x, t) = F(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x, 0) &= f(x) \quad \forall (x, t) \in \mathbb{R}^n \times \{t = 0\} \end{aligned} \quad (8)$$

where $F(x, t)$ is a source term. In order to get an integral equation for the problem with a potential $V(x, t)$, we will later replace the source term $F(x, t)$ by the potential term $V(x, t)u(x, t)$.

The initial-value problem for the nonhomogeneous Schrödinger equation (8) with nonhomogeneous initial conditions can be reduced to the analogous problem with homogeneous initial condition by using the integral representation

$$u(x, t) = \Phi(x, t) + \Pi(x, t) \quad (9)$$

where $u(x, t)$ is the solution of the nonhomogeneous problem, $\Phi(x, t)$ is the source term, and $\Pi(x, t)$ is the Poisson integral term (initial term). The Poisson integral term and the source term can be written as

$$\Pi(x, t) = e^{-itH/\hbar} f(x) = \int_{\mathbb{R}^n} K_f(x, y, t) f(y) dy \quad (10)$$

and

$$\Phi(x, t) = \int_0^t e^{-itH/\hbar} e^{i\tau H/\hbar} Lu(\cdot, \tau) d\tau = \int_0^t \int_{\mathbb{R}^n} K_f(x, y, t - \tau) Lu(y, \tau) dy d\tau \quad (11)$$

where $K_f(x, y, t)$ is the fundamental solution (free propagator) to the Schrödinger equation (3) in \mathbb{R}^n , and is given by

$$K_f(x, y, t) \equiv K_f(x - y, t) = \left(\frac{m}{2\pi\hbar it} \right)^{n/2} e^{im|x-y|^2/2\hbar t} \quad \forall x, y \in \mathbb{R}^n, t \neq 0 \quad (12)$$

The free propagator in this case can be expressed in the following manner $K_f(x, y, t - \tau)$. The free propagator K_f vanishes as a distribution as $t \rightarrow 0$ in the region $x \neq y$. In other words, the free propagator $K_f(x, y, t - \tau)$ satisfies the nonhomogeneous initial condition

$$K_f(x, y, 0) = \lim_{t \rightarrow \tau+} K_f(x, y, t - \tau) = \delta(x - y) \quad (13)$$

The following theorem introduces the Poisson integral, which is a solution of the Schrödinger equation without a potential term. Our discussion of the Poisson integral is somewhat more detailed than that of Evans [5], especially concerning the role of Abel summability.

Theorem 1 *Let $f(x)$ be a function on \mathbb{R}^n such that $(1 + |y|^2)f(y) \in L^1(\mathbb{R}^n)$. Then the Poisson integral*

$$u(x, t) = K_f * f = \int_{\mathbb{R}^n} K_f(x - y, t) f(y) dy \quad (14)$$

exists in the sense of Abel summability, and is a solution of the equation

$$Lu(x, t) = a^2 \Delta_x u(x, t) + i\hbar \partial_t u(x, t) = 0 \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}. \quad (15)$$

with given initial data. The Poisson integral defines a solution of the free Schrödinger equation in \mathbb{R}^n , $\forall t \neq 0$, even $t < 0$. This solution can be extended into $\mathbb{R}^n \times [0, \infty)$ with the initial condition $u(x, 0) = f(x)$ for all points x at which f is continuous.

Proof: If $|y|^2 f(y) \in L^1(\mathbb{R}^n)$, then the order of differentiation and integration in the equation (14) can be interchanged to show that the Poisson integral solves the Schrödinger equation. This hypothesis is obtained from Chapter IV of [5]. Let $y = x + \gamma z$, where $\gamma^2 = \frac{2\hbar t}{m}$; then we can rewrite the Poisson integral as

$$u(x, t) = \left(\frac{1}{\pi i}\right)^{n/2} \int_{\mathbb{R}^n} e^{i|z|^2} f(x + \gamma z) dz \quad (16)$$

where $|z| = \frac{|x-y|}{\gamma}$. Let ϵ be any positive number. Then

$$(\pi i)^{n/2} u(x, t) = \int_{\mathbb{R}^n} e^{i|z|^2} f(x + \gamma z) dz = I_1 + I_2 + I_3, \quad (17)$$

where

$$I_1 = \int_{|z| \leq \epsilon} e^{i|z|^2} \{f(x + \gamma z) - f(x)\} dz. \quad (18)$$

$$I_2 = \int_{|z| \geq \epsilon} e^{i|z|^2} f(x + \gamma z) dz \quad (19)$$

$$I_3 = \int_{|z| \leq \epsilon} e^{i|z|^2} f(x) dz \quad (20)$$

Now, we do some calculations for I_1 in hyperspherical coordinates $(\rho, \phi_1, \dots, \phi_{n-1})$. The continuity of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ at some point $y \in \mathbb{R}^n$ implies that $\forall \eta > 0 \exists \delta > 0$ such that $\forall x \in \mathbb{R}^n$ with $|x - y| < \delta$ implies that $|f(x) - f(y)| < \eta$, and x, y are points where f is continuous. Given ϵ , choose γ such that $\epsilon = \gamma\epsilon < \delta$, and let $\eta > 0$, then there exists a t so small such that $|f(x + \gamma z) - f(x)| < \eta$ for all z such that $|z| \leq \epsilon$. Therefore, by continuity, we obtain the following bounded estimate:

$$|I_1| \leq \eta \int_{|z| \leq \epsilon} dz \quad (21)$$

or

$$|I_1| \leq \eta \int_{|z| \leq \epsilon} dz \rightarrow 0 \quad (22)$$

as $\gamma|z| \rightarrow 0$ (i.e., $y \rightarrow x$) as $t \rightarrow 0$. Then since, $f \in L^1(\mathbb{R}^n)$

$$|I_2| \leq \int_{|z| \geq \epsilon} |f(x + \gamma z)| dz \rightarrow 0 \quad (23)$$

(not necessarily uniformly in x) as $\epsilon \rightarrow \infty$. In order to handle I_3 we use the Fresnel integral formula

$$\int_{\mathbb{R}^n} e^{i|z|^2} dz = (\pi i)^{n/2} \quad (24)$$

A proof of the one-dimensional Fresnel integral formula is outlined on [6]. The one-dimensional Fresnel integral implies the product version

$$\int_{\mathbb{R}^n} e^{i|z|^2} dz = \int_{\mathbb{R}^n} \exp\left(i \sum_{k=1}^n z_k^2\right) dz = \prod_{k=1}^n \int_{-\infty}^{\infty} e^{iz_k^2} dz_k = \prod_{k=1}^n (\pi i)^{1/2} = (\pi i)^{n/2} \quad (25)$$

Therefore, we have

$$\lim_{\epsilon \rightarrow \infty} I_2 = (\pi i)^{n/2} f(x). \quad (26)$$

Then we consider computing the Fresnel integral in terms of polar coordinates instead of Cartesian coordinates. Thus, we can rewrite the equation (25) by

$$\int_{\mathbb{R}^n} e^{i|z|^2} dz = \int_0^\infty \int_{\partial B(0,1)} e^{i\rho^2} \rho^{n-1} d\rho d\Omega = \omega_n \int_0^\infty \rho^{n-1} e^{i\rho^2} d\rho \quad (27)$$

Then we make use the substitution $t = \rho^2$, and hence we obtain

$$\int_{\mathbb{R}^n} e^{i|z|^2} dz = \frac{\omega_n}{2} \int_0^\infty t^{n-1} e^{it} dt \quad (28)$$

Then we insert the Abel factor $e^{-\alpha t}$ in the equation (28) and this gives

$$A(\alpha) = \frac{\omega_n}{2} \int_0^\infty e^{-\alpha t} t^{n-1} e^{it} dt \quad (29)$$

and ω_n is the surface area of the unit n -sphere. The surface area of the unit n -sphere is given by the following formula:

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \quad (30)$$

The complex Gaussian integral (also known as the Gaussian Fresnel integral) can be generalized for any positive integer, i.e., the Gaussian integral is a special case of $\int_0^\infty x^n e^{-x^2} dx$, when $n = 0$. The general Gaussian integral will be shown to be convergent when $n \in \mathbb{N}$: Consider the complex integral

$$P_n = \int_0^\infty s^{n-1} e^{is^2} ds \quad (31)$$

where $|z| = s = \frac{|x-y|}{\gamma} = \frac{\rho}{\gamma}$. Let $t = s^2$, and substituting this change of variables into equation (33) we have

$$P_n = \frac{1}{2} \int_0^\infty t^m e^{it} dt \quad (32)$$

where, $m = \frac{n-2}{2}$ and hence,

$$\int_0^\infty t^m e^{it} dt = \lim_{r \rightarrow \infty} \int_0^r t^m e^{it} dt \quad (33)$$

Once again, the change of variables $t = iz$ is performed and we have

$$\int_0^r t^m e^{it} dt = i \int_0^{-ir} (iz)^m e^{-z} dz \quad (34)$$

Then we insert the Abel factor $e^{-\alpha t}$ into the left-hand side of equation (34) and this gives

$$\int_0^r t^m e^{-\alpha t} e^{it} dt = i \int_0^{-ir} (iz)^m e^{-i\alpha z} e^{-z} dz \quad (35)$$

Then the path of integration is shifted from 0 to $-i\infty$, to 0 to ∞ . Therefore,

$$\lim_{r \rightarrow \infty} i \int_0^{-ir} (iz)^m e^{-i\alpha z} e^{-z} dz = \lim_{r \rightarrow \infty} i \int_0^r (iz)^m e^{-i\alpha z} e^{-z} dz \quad (36)$$

The integral in (36) is convergent for any positive $\alpha > 0$, and the integral over the semicircle at ∞ tends to 0. In order for the above limit to hold, the path of integration $(iz)^m$ must not go over a branch cut. Thus, if the integrand does not have any poles in the path of integration, then the integral in equation (33) is Abel summable and it is related to the gamma function $\Gamma(m+1)$. The integrand in equation (36) is an analytic function, and thus it does not have any poles or branch cuts in the path of integration. In other words,

$$\lim_{r \rightarrow \infty} i \int_0^{-ir} (iz)^m e^{-i\alpha z} e^{-z} dz = i^{m+1} \int_0^\infty z^m e^{-i\alpha z} e^{-z} dz \quad (37)$$

Then, we take the limit of $\alpha \rightarrow 0$, and the above equation becomes

$$\lim_{\alpha \rightarrow 0} i^{m+1} \int_0^\infty z^m e^{-i\alpha z} e^{-z} dz = i^{m+1} \int_0^\infty z^m e^{-z} dz \quad (38)$$

and this limit holds in the Abel sense, and we take $i^m = e^{i(\pi/2)m}$. Thus,

$$P_n = \frac{1}{2} i^{m+1} \Gamma(m+1) = \frac{i^{n/2}}{2} \Gamma\left(\frac{n}{2}\right) \quad (39)$$

and hence,

$$A(0) = \int_{\mathbb{R}^n} e^{i|z|^2} dz = \frac{i^{n/2} \omega_n}{2} \Gamma\left(\frac{n}{2}\right) \quad (40)$$

This confirms equation (25) in an alternative way.

This implies the continuity of $u(x, t)$ at $t = 0$. Therefore, the Poisson integral has the initial values $u(\cdot, 0) = f(x)$ for all points x at which f is continuous. \square

The proof of theorem 2 is partly based on [7]. In [7], the authors construct an abstract lemma for a function $\psi(t)$ that belongs to the domain of the operator H , and approximately solves the Schrödinger equation in the following sense

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H\psi(t) + F(t) \quad (41)$$

where $F(t)$ is a source term. In our case, we want to prove that there exists a function $u(t)$ that is the exact solution to the same Schrödinger problem with the difference that $H = H_0 = -a^2 \Delta$. The other difference is that we are trying to prove that the solution can be expressed a Volterra integral equation of the second kind.

All throughout this article the letter I will denote the time interval $(0, T)$, where T is a positive constant. In theorem 2, we will define the $L^\infty(I; \mathcal{B})$ norm of a function u to be

$$\|u\|_{L^\infty(I; \mathcal{B})} = \inf\{M \geq 0 : \|u(t)\|_{\mathcal{B}} \leq M, \text{ holds for almost all } t \in [0, T]\} \quad (42)$$

where \mathcal{B} is a Banach space. For a more formal and general definition of the L^∞ norm look at definitions (4)-(5).

Theorem 2 *Let $f(x)$ be a function on \mathbb{R}^n with the following property: $(1 + |y|^2)f(y) \in L^1(\mathbb{R}^n)$. Furthermore, suppose the source term $F(x, t)$ satisfies the condition*

$$\|F(\cdot, t)\| \leq \xi(t) \quad (43)$$

where $\|\cdot\|$ is a Banach space norm and the function $\xi(t)$ satisfies the following condition

$$\|\xi\|_{L^\infty(I)} \leq M \quad (44)$$

where M and T are positive constants. The solution of the initial-value problem for the nonhomogeneous Schrödinger equation (8) can be represented as the following integral formula:

$$u(x, t) = \Phi(x, t) + \Pi(x, t) \quad (45)$$

The initial term, and the source term are given by the following integral representation formulas:

$$\Pi(x, t) = \int_{\mathbb{R}^n} K_f(x, y, t) f(y) dy \quad (46)$$

and

$$\Phi(x, t) = -\frac{i}{\hbar} \int_0^t \int_{\mathbb{R}^n} K_f(x, t; y, \tau) F(y, \tau) dy d\tau \quad (47)$$

where, $K_f(x, t; y, \tau)$ is the fundamental solution and $u(x, 0) = f(x)$. The solution u will belong to the Banach space $L^\infty(I; \mathcal{B})$.

Proof: Suppose that $f(x)$ be a function on \mathbb{R}^n with the following property: $(1 + |y|^2)f(y) \in L^1(\mathbb{R}^n)$. Let us assume $t > 0$. The Poisson integral term was shown to solve the initial-value problem for the homogeneous Schrödinger equation. Let $u(x, t)$ be the solution to the nonhomogeneous Schrödinger equation (8). Then we claim that solution $u(x, t)$ can be expressed as a Volterra integral representation

$$u(x, t) = \int_{\mathbb{R}^n} K_f(x, y, t) f(y) dy - \frac{i}{\hbar} \int_0^t \int_{\mathbb{R}^n} K_f(x, t; y, \tau) F(y, \tau) dy d\tau \quad (48)$$

In other words, we claim that the solution for the nonhomogeneous Schrödinger equation is given by

$$u(x, t) = u_h(x, t) + u_p(x, t) \quad (49)$$

where u_h is the solution of the homogeneous Schrödinger equation, and u_p is the particular solution of the nonhomogeneous Schrödinger equation. Then by the Poisson integral theorem, the function u_h is the solution of the homogeneous Schrödinger equation and hence $Lu_h = 0$, $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$. Then by applying the Schrödinger operator to $u(t)$, we have

$$\begin{aligned} Lu &= Lu_p = -a^2 \Delta_x u_p + i\hbar \frac{\partial u_p}{\partial t} = \\ &= -a^2 \int_{\mathbb{R}^n} \Delta_x K_f(x, y, t) f(y) dy + i\hbar \frac{\partial}{\partial t} \left(-\frac{i}{\hbar} \int_0^t \int_{\mathbb{R}^n} K_f(x, y, t - \tau) F(y, \tau) dy d\tau \right) \end{aligned} \quad (50)$$

but $\Delta_x K_f(x, y, t - \tau) = 0$, $\forall t > \tau$, and hence

$$\begin{aligned} Lu &= Lu_p = \frac{\partial}{\partial t} \left(\int_0^t \int_{\mathbb{R}^n} K_f(x, y, t - \tau) F(y, \tau) dy d\tau \right) \\ &= \int_{\mathbb{R}^n} K_f(x, y, t - t) F(y, t) dy = \int_{\mathbb{R}^n} K_f(x, y, 0) F(y, t) dy \\ &= \int_{\mathbb{R}^n} \delta(x - y) F(y, t) dy = F(x, t) \end{aligned} \quad (51)$$

and hence the solution $u(t)$ is given by the Volterra integral (48).

Now, this equation should satisfy the initial condition $u(x, 0) = f(x)$. This implies that the second term should vanish as $t \rightarrow 0$.

Another way to express equation (48) is via unitary operators. Hence,

$$e^{-itH/\hbar} f(x) - \frac{i}{\hbar} \int_0^t e^{-itH/\hbar} e^{i\tau H/\hbar} F(\tau) d\tau = u(t) \quad (52)$$

Let us consider the following integral,

$$-i\hbar^{-1} \int_0^t e^{i\tau H/\hbar} Lu(\tau) d\tau \quad (53)$$

and since $Lu = -a\Delta u + i\hbar \frac{\partial u}{\partial \tau}$, we can rewrite the above equation in the following way

$$\begin{aligned} -i\hbar^{-1} \int_0^t e^{i\tau H/\hbar} Lu(\tau) d\tau &= -i\hbar^{-1} \int_0^t e^{i\tau H/\hbar} \left(-a\Delta u(\tau) + i\hbar \frac{\partial u(\tau)}{\partial \tau} \right) d\tau \\ &= -i\hbar^{-1} \int_0^t e^{i\tau H/\hbar} \left(-Hu(\tau) + i\hbar \frac{\partial u}{\partial \tau} \right) d\tau \\ &= \int_0^t \frac{\partial}{\partial \tau} \left(-e^{i\tau H/\hbar} u(\tau) \right) d\tau = -e^{itH/\hbar} u(t) + u(0) \\ &= u(0) - e^{itH/\hbar} u(t) \end{aligned} \quad (54)$$

Then, we can rewrite the above equation into the following form:

$$u(t) = e^{-itH/\hbar} u(0) - i\hbar^{-1} e^{-itH/\hbar} \int_0^t e^{i\tau H/\hbar} Lu(\tau) d\tau \quad (55)$$

and this is the Volterra integral equation (48) written in unitary operator notation. Let us take expression $e^{-itH/\hbar} u(0) - u(t)$ into consideration. This expression is simply the source term $\Phi(x, t)$. Then we take a Banach space norm to the quantity $e^{-itH/\hbar} u(0) - u(t)$, and by the unitarity of the propagator and the fundamental theorem of calculus, we have

$$\begin{aligned} \|\Phi(t)\| &= \|e^{-itH/\hbar} u(0) - u(t)\| = \left\| i\hbar^{-1} e^{itH/\hbar} \int_0^t e^{i\tau H/\hbar} Lu(\tau) d\tau \right\| \\ &\leq \hbar^{-1} \int_0^t \|Lu\| d\tau \leq \hbar^{-1} \int_0^t \xi(\tau) d\tau \end{aligned} \quad (56)$$

since by hypothesis we have that $\|Lu\| \leq \xi(\tau)$. Hence,

$$\begin{aligned} \|\Phi(t)\| &= \|e^{-itH/\hbar} u(0) - u(t)\| \leq \hbar^{-1} \int_0^t \|\xi\|_{L^\infty(I)} d\tau \\ &\leq \hbar^{-1} \|\xi\|_{L^\infty(I)} t \end{aligned} \quad (57)$$

Then by taking the limit of $t \rightarrow 0$, the above inequality also goes to 0. The Poisson integral term $\Pi(x, t)$ satisfies the initial condition by theorem 1. Therefore, the Volterra integral equation (48) satisfies the initial condition and is a solution to the nonhomogeneous Schrödinger equation. \square

3. Integral Equations and Neumann Series

In this section, we introduce the integral operators in arbitrary Banach spaces in order to find a solution to the Schrödinger equation in \mathbb{R}^{n+1} . This section is an informal preview of the Volterra and General Volterra Theorems which will be proved in section 4. In the following analysis of integral operators, this article will use as a foundation Kress's treatment of linear integral equations [3]. In operator notation, the Volterra integral equation of the second kind is written in the following manner:

$$\phi - \hat{Q}\phi = f \quad (58)$$

where it is assumed that \hat{Q} is a bounded linear operator from a Banach space \mathcal{B} to itself and $\phi, f \in \mathcal{B}$. The existence and uniqueness of a solution to an integral operator equation can be found via the inverse operator $(I - \hat{Q})^{-1}$, where I is the identity operator. The existence of the inverse operator will become clear below.

Definition 3 Let $BL(\mathcal{B}; \mathcal{B})$ be the collection of bounded linear transformations from \mathcal{B} into \mathcal{B} . Also, we denote the space $BL(\mathcal{B}, \mathbb{F})$ as the set of bounded linear functionals on \mathcal{B} , where $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$.

Important Banach spaces which we will be dealing with are the Lebesgue spaces $L^p(\mu)$. This article will cover the case when $p = \infty$, in order to create bounded estimates of the Volterra operator \hat{Q} with respect to the norm $\|\cdot\|$. In the next section, the Volterra Theorem will prove that the spectral radius of the Volterra operator is zero using the L^∞ -estimates.

Definition 4 Let (Ω, Σ, μ) be a measure space and \mathcal{B} be a Banach space. The collection of all essentially bounded measurable functions on Ω taking values in \mathcal{B} is denoted $L^\infty(\Omega, \mu; \mathcal{B})$, and the reference to μ is omitted when there is no danger of confusion. The essential supremum of a function $\varphi: \Omega \rightarrow \mathcal{B}$ is given by

$$\|\varphi\|_{L^\infty(\Omega; \mathcal{B})} = \inf\{M \geq 0 : \|\varphi(x)\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq M \text{ holds for almost all } x\}. \quad (59)$$

If φ does have an essential bound, then it is said to belong to $L^\infty(\Omega; \mathcal{B})$.

Definition 5 Let \mathcal{B}_1 and \mathcal{B}_2 be Banach spaces and Ω be some measurable space. Suppose that the function $A(\cdot, \cdot)$ is measurable, and define its norm

$$\|A(x, y)\|_{\mathcal{B}_1 \rightarrow \mathcal{B}_2} = \inf\{M \geq 0 : \|A(x, y)\phi\| \leq M\|\phi\|, \forall \phi \in \mathcal{B}_1\} \quad (60)$$

If $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$, then we use the following notation for the Banach space norm of $A(x, y)$:

$$\|A(x, y)\| = \|A(x, y)\|_{\mathcal{B}} = \inf\{M \geq 0 : \|A(x, y)\phi\| \leq M\|\phi\|, \forall \phi \in \mathcal{B}\} \quad (61)$$

For each $(x, y) \in \Omega \times \Omega$ let $A(x, y) : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a bounded linear operator with norm

$$\begin{aligned} \|A\|_{L^\infty(\Omega^2; \mathcal{B}_1 \rightarrow \mathcal{B}_2)} &\equiv \inf\{M \geq 0 : \|A(x, y)\| \leq M, \text{ for almost all } (x, y) \in \Omega^2\} \\ &= \text{ess sup}_{(x, y) \in \Omega^2} \|A(x, y)\|_{\mathcal{B}_1 \rightarrow \mathcal{B}_2} \end{aligned} \quad (62)$$

and $A(\cdot, \cdot)$ is called a bounded operator kernel if $\|A\|_{L^\infty(\Omega^2; \mathcal{B}_1 \rightarrow \mathcal{B}_2)}$ is finite.

In Section 2 we have reformulated the Schrödinger equation as an integral equation of the second kind. The existence and uniqueness of its solution can be found by analysis of the Neumann series. The successive approximations

$$\phi_{n+1} = \hat{Q}\phi_n + f \quad (63)$$

converge to the exact solution of the integral equation (58), if some technical conditions are satisfied. Otherwise, the successive approximations may not converge even for the Volterra operator, e.g. if its kernel is not good enough or the interval is infinite.

The above integral equations are given for an arbitrary Banach space \mathcal{B} that will be used in Picard's algorithm of successive approximation. Then, equation (63) is converging to the solution ϕ if the following conditions are satisfied:

- 1) the integral operator \hat{Q} is a bounded linear operator in the Banach space \mathcal{B} .
- 2) the function f belongs to a Banach space \mathcal{B} ,
- 3) and finally, the infinite series $\varphi = \sum_{j=0}^{\infty} \hat{Q}^j f$ is a convergent series with respect to the topology of L^∞ in time and of \mathcal{B} in space.

If these three conditions are satisfied, then it is possible to use the Neumann series to obtain the exact solution to the original problem, which is the initial value problem of the Schrödinger equation with a potential term $V(x, t)$. The three conditions turn out to be the necessary hypotheses to prove the Volterra and L^p Volterra theorems.

4. Volterra Kernels and Successive Approximations

In this section we will revisit the method of successive approximations. Our main theorem will prove the well-known fact that the Volterra integral operator of the second kind has a spectral radius of zero. This direct proof is not found in [3], where the logic runs in the other direction. That is, our proof of the Volterra and General Volterra theorems will not use the spectral radius to prove that the Neumann series converges. The Volterra operator is known to have a nice property, known as the simplex structure. It is the simplex structure which makes the infinite Neumann series converge. Then it follows from the convergence of the Neumann series that the spectral radius is zero.

Now, we focus our attention to four hypotheses which will be used in lemma 6 and lemma 7. Lemma 6 will be inductively applied in proving the L^∞ Volterra theorem. Similarly, lemma 8 will also be inductively applied to the L^p Volterra theorem. The following are the four hypotheses that are needed for lemmas 6, 7, and 8 :

- 1.) Let \mathcal{B} be a Banach space, and let I be an interval.
- 2.) Assume that $\forall (t, \tau) \in \bar{I}^2, A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$.
- 3.) Suppose that the kernel $A(t, \tau)$ is a measurable and uniformly bounded linear

operator (in the sense of definition 6) .

4.) Finally, let $A(t, \tau)$ satisfy the Volterra condition, $A(t, \tau) = 0$ if $\tau > t$.

In theorem 9, we have that $\Omega = I = (0, T)$, and the Lebesgue space is $L^\infty(I; \mathcal{B})$, i.e. $p = \infty$. And for theorem 10, $\Omega = I$, and the Lebesgue space we consider is $L^1(I; \mathcal{B})$, where $p = 1$. Lemma 6 and lemma 7 are needed to conclude that $\|\hat{Q}\phi\|_{L^p(I; \mathcal{B})} \leq Dt\|\phi\|_{L^p(I; \mathcal{B})}$ for $D > 0$ and $t \in [0, T]$, where $p = 1$ and $p = \infty$. Finally, lemma 8 is needed in theorem 11 to conclude that $\|\hat{Q}\phi\|_{L^p(I; \mathcal{B})} \leq M(p)t\|\phi\|_{L^p(I; \mathcal{B})}$ where $M(p) = \frac{D}{p^{1/p}}$, $t \in [0, T]$ and $1 < p < \infty$. In the lemmas 6-8, n is any real positive number.

Lemma 6 Let the Volterra integral operator, $\hat{Q} : L^\infty(I; \mathcal{B}) \rightarrow L^\infty(I; \mathcal{B})$, be defined by

$$\hat{Q}\phi(t) = \int_0^T A(t, \tau)\phi(\tau) d\tau = \int_0^t A(t, \tau)\phi(\tau) d\tau, \quad (64)$$

where $\phi \in \mathcal{B}$. Let $\phi \in L^\infty(I; \mathcal{B})$, and assume that $\exists C > 0$, for each subinterval J of the form $(0, t)$, such that $\|\phi\|_{L^\infty(J; \mathcal{B})} = \sup_{0 \leq \tau < t} \|\phi(\tau)\| \leq Ct^n$. Assume that hypotheses (1)-(4) are satisfied. Then it follows that $\|\hat{Q}\phi\|_{L^\infty(J; \mathcal{B})} \leq \frac{DCt^{n+1}}{n+1}$.

Proof: Since $A(t, \tau)$ satisfies definition 6, then $\exists D > 0$ such that $\forall (t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| \leq D < \infty$. The $L^\infty(J; \mathcal{B})$ norm of the function $\hat{Q}\phi(t)$ is

$$\begin{aligned} \|\hat{Q}\phi\|_{L^\infty(I; \mathcal{B})} &= \sup_{t_1 \leq t} \left\| \int_0^{t_1} A(t, \tau)\phi(\tau) d\tau \right\| \leq \sup_{t_1 \leq t} \int_0^{t_1} \|A(t, \tau)\phi(\tau)\| d\tau \\ &\leq \sup_{t_1 \leq t} \int_0^{t_1} \|A(t, \tau)\| \|\phi(\tau)\| d\tau \leq \sup_{t_1 \leq t} \int_0^{t_1} DC\tau^n d\tau \\ &= \sup_{t_1 \leq t} DC \frac{t_1^{n+1}}{n+1} = \frac{DCt^{n+1}}{n+1} \end{aligned} \quad (65)$$

□

Lemma 7 Let the Volterra integral operator, $\hat{Q} : L^1(I; \mathcal{B}) \rightarrow L^1(I; \mathcal{B})$, be defined by

$$\hat{Q}\phi(t) = \int_0^T A(t, \tau)\phi(\tau) d\tau = \int_0^t A(t, \tau)\phi(\tau) d\tau, \quad (66)$$

where $\phi \in \mathcal{B}$. Let $\phi \in L^1(I; \mathcal{B})$, and assume that $\exists C > 0$, for each subinterval J of the form $(0, t)$, such that

$$\|\phi\|_{L^1(J; \mathcal{B})} = \int_0^t \|\phi(\tau)\| d\tau \leq Ct^n \quad (67)$$

Assume that hypotheses (1)-(4) are satisfied. Then it follows that $\|\hat{Q}\phi\|_{L^1(J; \mathcal{B})} \leq \frac{DCt^{n+1}}{n+1}$.

Proof: Since $A(t, \tau)$ satisfies definition 6, then $\exists D > 0$, such that $\forall (t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| \leq D < \infty$. The $L^1(J; \mathcal{B})$ norm of the function $\hat{Q}\phi(t)$ is

$$\begin{aligned} \|\hat{Q}\phi\|_{L^1(J; \mathcal{B})} &= \int_0^t \left\| \int_0^{t_1} A(t, \tau)\phi(\tau) d\tau \right\| dt_1 \leq \int_0^t \int_0^{t_1} \|A(t, \tau)\| \|\phi(\tau)\| d\tau \\ &\leq \int_0^t D \int_0^{t_1} \|\phi(\tau)\| d\tau dt_1 \leq \int_0^t D \|\phi\|_{L^1(J; \mathcal{B})} dt_1 \\ &\leq \int_0^t DCt_1^n dt_1 = \frac{DCt^{n+1}}{n+1} \end{aligned} \quad (68)$$

□

The proof of lemma 8 uses Folland's proof of Young's inequality as a model[9].

Lemma 8 *Let the Volterra integral operator, $\hat{Q} : L^p(I; \mathcal{B}) \rightarrow L^p(I; \mathcal{B})$, be defined by*

$$\hat{Q}\phi(t) = \int_0^T A(t, \tau)\phi(\tau) d\tau = \int_0^t A(t, \tau)\phi(\tau) d\tau, \quad (69)$$

where $\phi \in \mathcal{B}$. Let $\phi \in L^p(I; \mathcal{B})$, where $1 < p < \infty$, and assume that for each subinterval J , $\exists C > 0$ such that

$$\|\phi\|_{L^p(J; \mathcal{B})} = \left(\int_0^t \|\phi(\tau)\|^p d\tau \right)^{1/p} \leq Ct^n. \quad (70)$$

Assume that hypotheses (1)-(4) are satisfied. Then it follows that $\|\hat{Q}\phi\|_{L^p(J; \mathcal{B})} \leq CD \frac{t^{n+1}}{[p(n+1)]^{1/p}}$.

Proof: Suppose $1 < p < \infty$, and let q be the conjugate exponent ($p^{-1} + q^{-1} = 1$). Since $A(t, \tau)$ satisfies definition 6, then $\exists D > 0$, such that $\forall (t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| \leq D < \infty$. The Banach space norm of the function $\hat{Q}\phi(t)$ is

$$\begin{aligned} \|\hat{Q}\phi(t_1)\| &\leq \left(\int_0^{t_1} \|A(t_1, \tau)\| d\tau \right)^{1/q} \left(\int_0^{t_1} \|A(t_1, \tau)\| \|\phi(\tau)\|^p d\tau \right)^{1/p} \\ &\leq D^{1/q} \left(\int_0^{t_1} d\tau \right)^{1/q} \left(\int_0^{t_1} D \|\phi(\tau)\|^p d\tau \right)^{1/p} \leq D^{1/q} D^{1/p} t_1^{1/q} \left(\int_0^{t_1} \|\phi(\tau)\|^p d\tau \right)^{1/p} \\ &\leq D t_1^{1/q} \left(\int_0^{t_1} \|\phi(\tau)\|^p d\tau \right)^{1/p} \end{aligned} \quad (71)$$

and then we raise both sides to the p th power and integrating, we see by Fubini's theorem that

$$\begin{aligned} \int_0^{t_1} \|\hat{Q}\phi(t_1)\|^p dt_1 &\leq \int_0^t D^p t_1^{p/q} \int_0^{t_1} \|\phi(\tau)\|^p d\tau dt_1 \leq \int_0^t D^p \int_0^t t_1^{p/q} \|\phi\|_{L^p(J; \mathcal{B})}^p dt_1 \\ &\leq D^p \int_0^t C t_1^{np+p/q} dt_1 \leq D^p C^p t^{p/q} \frac{t^{np+p/q+1}}{np + \frac{p}{q} + 1} \\ &\leq D^p C^p \frac{t^{np+p/q+1}}{np + p} \end{aligned} \quad (72)$$

since $\frac{1}{q} = 1 - \frac{1}{p}$, and hence $\frac{p}{q} = p - 1$. Now we take the p -th root, and this yields

$$\|\hat{Q}\phi\|_{L^p(J; \mathcal{B})} \leq DC \frac{t^{n+1/p+1/q}}{[p(n+1)]^{1/p}} = DC \frac{t^{n+1}}{[p(n+1)]^{1/p}} = DC \frac{t^{n+1}}{[p(n+1)]^{1/p}} \quad (73)$$

□

The following theorem is the main theorem of [1]. It is also used in the examples of section 4. The General Volterra Theorem is just a variant of the Volterra Theorem, i.e., it is the L^p -analogue.

Theorem 9 (*L^∞ Volterra Theorem*) *Let $I = (0, T)$ such that $0 < T < \infty$. Assume that the kernel $A(t, \tau)$ is a measurable and uniformly bounded linear operator. Furthermore, assume that $\forall (t, \tau) \in \bar{I}^2$, $A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$ where \mathcal{B} is a Banach space. Suppose that the*

kernel satisfies the following condition, $A(t, \tau) = 0$, when $\tau > t$. The Volterra integral operator, $\hat{Q} : L^\infty(I; \mathcal{B}) \rightarrow L^\infty(I; \mathcal{B})$, is defined by

$$\hat{Q}\varphi(t) = \int_0^T A(t, \tau)\varphi(\tau) d\tau = \int_0^t A(t, \tau)\varphi(\tau) d\tau, \quad (74)$$

where $\varphi \in \mathcal{B}$. Then, the Volterra integral equation with the kernel $A(t, \tau)$ can be solved by successive approximations. That is, the Neumann series converges in the topology of $L^\infty(I; \mathcal{B})$.

Proof: Let $L^\infty(J; \mathcal{B})$ be the Banach space with norm $\|\cdot\|_{L^\infty(J; \mathcal{B})}$, where $J = (0, t)$ and where $0 < t \leq T$. Assume that hypotheses (1)-(4) are satisfied. Then by definition 6, $\exists D > 0$ such that $\forall(t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| = D < \infty$.

Suppose the function $\psi_0 : J \rightarrow L^\infty(J; \mathcal{B})$ is a bounded function with norm $\|\psi_0\|_{L^\infty(J; \mathcal{B})} = \sup_{0 < \tau < t} \|\psi_0(\tau)\|$. Then by lemma 6, it follows that the norm of $\psi_0(t)$ satisfies the following condition, $\|\psi_0\|_{L^\infty(J; \mathcal{B})} \leq Nt^0 = N$, since $n = 0$. Then, we compute L^∞ norm estimates for the following equation:

$$\begin{aligned} \|\psi_1\|_{L^\infty(J; \mathcal{B})} &= \|\hat{Q}\psi_0\|_{L^\infty(J; \mathcal{B})} = \sup_{t_1 \leq t} \left\| \int_0^{t_1} A(t_1, \tau)\psi_0(\tau) d\tau \right\| \\ &\leq \sup_{t_1 \leq t} \int_0^{t_1} \|A(t, \tau)\| \|\psi_0(\tau)\| d\tau \leq \sup_{t_1 \leq t} \int_0^{t_1} DN d\tau = D N t \end{aligned}$$

and if $N = \|\psi_0\|_{L^\infty(J; \mathcal{B})}$, then it follows that

$$\|\hat{Q}\psi_0\|_{L^\infty(J; \mathcal{B})} \leq D\|\psi_0\|_{L^\infty(J; \mathcal{B})}t. \quad (75)$$

Then by inductively applying lemma 6, we see that the n th term of the Neumann series ψ_n gives the simplex structure:

$$\|\psi_n\|_{L^\infty(J; \mathcal{B})} \leq D^n \|\psi_0\|_{L^\infty(J; \mathcal{B})} \frac{t^n}{n!} \quad (76)$$

Therefore the series $\sum_{n=0}^\infty \psi_n$ is majorized by

$$\begin{aligned} \|\psi_0\|_{L^\infty(J; \mathcal{B})} + \|\psi_0\|_{L^\infty(J; \mathcal{B})} \sum_{n=1}^\infty \frac{D^n t^n}{n!} &= \|\psi_0\|_{L^\infty(J; \mathcal{B})} \sum_{n=0}^\infty \frac{D^n t^n}{n!} \\ &= \|\psi_0\|_{L^\infty(J; \mathcal{B})} e^{Dt}. \end{aligned} \quad (77)$$

and the above estimate holds for all $t \in (0, T]$. Therefore, the Neumann series converges in the topology of $L^\infty(I; \mathcal{B})$. \square

Theorem 10 (L^1 Volterra Theorem) Let $I = (0, T)$ such that $0 < T < \infty$. Assume that the kernel $A(t, \tau)$ is a measurable and uniformly bounded linear operator (in the sense of definition 6). Furthermore, assume that $\forall(t, \tau) \in \bar{I}^2$, $A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$ where \mathcal{B} is a Banach space. Suppose that the kernel satisfies the following condition, $A(t, \tau) = 0$, when $\tau > t$. The Volterra integral operator, $\hat{Q} : L^1(I; \mathcal{B}) \rightarrow L^1(I; \mathcal{B})$, is defined by

$$\hat{Q}\varphi(t) = \int_0^T A(t, \tau)\varphi(\tau) d\tau = \int_0^t A(t, \tau)\varphi(\tau) d\tau, \quad (78)$$

where $\varphi \in \mathcal{B}$. Then, the Volterra integral equation with the kernel $A(t, \tau)$ can be solved by successive approximations. That is, the Neumann series converges in the topology of $L^1(I; \mathcal{B})$.

Proof: Let $L^1(J; \mathcal{B})$ be the Banach space with norm $\|\cdot\|_{L^1(J; \mathcal{B})}$, where $J = (0, t)$ and where $0 < t \leq T < \infty$. Suppose the function $\psi_0 : J \rightarrow L^1(J; \mathcal{B})$ is a bounded function with norm

$$\|\psi_0\|_{L^1(J; \mathcal{B})} = \int_0^t \|\psi_0(\tau)\| d\tau \quad (79)$$

Define the Volterra integral operator in the following way,

$$\hat{Q}\phi(t) = \int_0^t A(t, \tau)\phi(\tau) d\tau, \quad (80)$$

where $\phi \in \mathcal{B}$. Suppose that $\forall(t, \tau) \in \bar{I}^2$, $A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$ is a measurable and uniformly bounded operator. Then by definition 6, $\exists D > 0$ such that $\forall(t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| = D < \infty$. Furthermore, $A(t, \tau) = 0$ when $\tau > t$. Then by lemma 8, it follows that the norm of $\psi_0(t)$ satisfies the condition $\|\psi_0\|_{L^1(J; \mathcal{B})} \leq Nt^0 = N$, since $n = 0$. Then, we compute L^1 norm estimates for the following equation:

$$\begin{aligned} \|\psi_1\|_{L^1(J; \mathcal{B})} &= \|\hat{Q}\psi_0\|_{L^1(J; \mathcal{B})} = \int_0^t \left\| \int_0^{t_1} A(t_1, \tau)\psi_0(\tau) d\tau \right\| dt_1 \\ &\leq \int_0^t \int_0^{t_1} \|A(t_1, \tau)\| \|\psi_0(\tau)\| d\tau dt_1 \leq \int_0^t \int_0^{t_1} D\|\psi_0(\tau)\| d\tau dt_1 \\ &\leq \int_0^t D\|\psi_0\|_{L^1(J; \mathcal{B})} dt_1 \leq DN \int_0^t dt_1 = D N t \end{aligned} \quad (81)$$

where $t_1 \leq t$ and if $N = \|\psi_0\|_{L^1(J; \mathcal{B})}$, then it follows that

$$\|\hat{Q}\psi_0\|_{L^1(J; \mathcal{B})} \leq D\|\psi_0\|_{L^1(J; \mathcal{B})}t. \quad (82)$$

Then by inductively applying lemma 6, we see that the n th term of the Neumann series ψ_n gives the simplex structure:

$$\|\psi_n\|_{L^1(J; \mathcal{B})} \leq D^n \|\psi_0\|_{L^1(J; \mathcal{B})} \frac{t^n}{n!} \quad (83)$$

Thus the series $\sum_{n=0}^{\infty} \psi_n$ is majorized by

$$\|\psi_0\|_{L^1(J; \mathcal{B})} + \|\psi_0\|_{L^1(J; \mathcal{B})} \sum_{n=1}^{\infty} D^n \frac{t^n}{n} = \|\psi_0\|_{L^1(J; \mathcal{B})} e^{Dt} \quad (84)$$

and the above estimate holds for all $t \in (0, T]$. Therefore, the Neumann series converges with respect to the topology $L^1(I; \mathcal{B})$. \square

Theorem 11 (*L^p Volterra Theorem*) Let $I = (0, T)$ such that $0 < T \leq \infty$. Assume that the kernel $A(t, \tau)$ is a measurable and uniformly bounded linear operator (in the sense of definition 6). Furthermore, assume that $\forall(t, \tau) \in \bar{I}^2$, $A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$ where \mathcal{B} is a Banach space. Suppose that the kernel satisfies the following condition, $A(t, \tau) = 0$, when $\tau > t$. The Volterra integral operator, $\hat{Q} : L^p(I; \mathcal{B}) \rightarrow L^p(I; \mathcal{B})$, is defined by

$$\hat{Q}\varphi(t) = \int_0^T A(t, \tau)\varphi(\tau) d\tau = \int_0^t A(t, \tau)\varphi(\tau) d\tau, \quad (85)$$

where $\varphi \in \mathcal{B}$. Then, the Volterra integral equation with the above kernel $A(t, \tau)$ can be solved by successive approximations. That is, the Neumann series converges in the topology of $L^p(I; \mathcal{B})$.

Proof: Let $L^p(J; \mathcal{B})$ be the Banach space with norm $\|\cdot\|_{L^p(J; \mathcal{B})}$, where $J = (0, t)$ and where $0 < t \leq T < \infty$. Suppose the function $\psi_0 : J \rightarrow L^p(J; \mathcal{B})$ is a bounded function with norm

$$\|\psi_0\|_{L^p(J; \mathcal{B})} = \left(\int_0^t \|\psi_0(\tau)\|^p d\tau \right)^{1/p} \quad (86)$$

Define the Volterra integral operator in the following way,

$$\hat{Q}\phi(t) = \int_0^t A(t, \tau)\phi(\tau) d\tau, \quad (87)$$

where $\phi \in \mathcal{B}$. Suppose that $\forall(t, \tau) \in \bar{I}^2$, $A(t, \tau) : \mathcal{B} \rightarrow \mathcal{B}$ is a measurable and uniformly bounded operator. Then by definition 6, $\exists D > 0$ such that $\forall(t, \tau) \in \bar{I}^2$, $\|A(t, \tau)\| = D < \infty$. Furthermore, $A(t, \tau) = 0$ when $\tau > t$. Then by lemma 7, it follows that the norm of $\psi_0(t)$ satisfies the following condition, $\|\psi_0\|_{L^p(J; \mathcal{B})} \leq Nt^0 = N$, since $n = 0$. The Banach space norm of the function $\hat{Q}\psi_0(t)$ is

$$\begin{aligned} \|\hat{Q}\psi_0(t_1)\| &\leq \left(\int_0^{t_1} \|A(t_1, \tau)\| d\tau \right)^{1/q} \left(\int_0^{t_1} \|A(t_1, \tau)\| \|\psi_0(\tau)\|^p d\tau \right)^{1/p} \\ &\leq D^{1/q} \left(\int_0^{t_1} d\tau \right)^{1/q} \left(\int_0^{t_1} D \|\psi_0(\tau)\|^p d\tau \right)^{1/p} \\ &\leq D^{1/q} D^{1/p} t_1^{1/q} \left(\int_0^{t_1} \|\psi_0(\tau)\|^p d\tau \right)^{1/p} \leq D t_1^{1/q} \left(\int_0^{t_1} \|\psi_0(\tau)\|^p d\tau \right)^{1/p}. \end{aligned} \quad (88)$$

Raising both sides to the p th power and integrating, we see by Fubini's theorem that

$$\begin{aligned} \int_0^{t_1} \|\hat{Q}\psi_0(t_1)\|^p dt_1 &\leq \int_0^t D^p t_1^{p/q} \int_0^{t_1} \|\psi_0(\tau)\|^p d\tau dt_1 \leq D^p \int_0^t t_1^{p/q} \|\psi_0\|_{L^p(J; \mathcal{B})}^p dt_1 \\ &\leq D^p \int_0^t N^p t_1^{p/q} dt_1 \leq D^p N^p \frac{t^{p/q+1}}{\frac{p}{q}+1} \\ &\leq D^p N^p \frac{t^{p/q+1}}{p} \end{aligned} \quad (89)$$

where $t_1 \leq t$. Now, taking the p th root, we obtain

$$\|\hat{Q}\psi_0\|_{L^p(J; \mathcal{B})} \leq DN \frac{t^{1/q+1/p}}{p^{1/p}} = \frac{DN}{p^{1/p}} t \quad (90)$$

and if $N = \|\psi_0\|_{L^p(J; \mathcal{B})}$, then it follows that

$$\|\hat{Q}\psi_0\|_{L^p(J; \mathcal{B})} \leq \frac{D}{p^{1/p}} \|\psi_0\|_{L^p(J; \mathcal{B})} t \quad (91)$$

converges with respect to the norm $\|\cdot\|_{L^p}$. Then by inductively applying lemma 7, we see that the n th term of the Neumann series ψ_n gives the following result:

$$\|\psi_n\|_{L^p(J; \mathcal{B})} \leq \frac{D^n}{p^{n/p}} \|\psi_0\|_{L^p(J; \mathcal{B})} \frac{t^n}{(n!)^{1/p}} \quad (92)$$

Thus the series $\sum_{n=0}^{\infty} \psi_n$ is majorized by

$$\|\psi_0\|_{L^p(J;\mathcal{B})} + \|\psi_0\|_{L^p(J;\mathcal{B})} \sum_{n=1}^{\infty} \frac{D^n}{p^{n/p}} \frac{t^n}{(n!)^{1/p}} \quad (93)$$

Now, we investigate whether the Neumann series are convergent. The ratio test will be used to determine whether the infinite series converges or diverges. Let L be defined in the following manner,

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad (94)$$

In this case, $a_n = \|\psi_n\|_{L^p(J;\mathcal{B})}$, and $a_{n+1} = \|\psi_{n+1}\|_{L^p(J;\mathcal{B})}$. Since $\|\psi_n\|_{L^p(J;\mathcal{B})}$ is always a positive number, the absolute values in equation (94) can be removed and we obtain:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{\|\psi_{n+1}\|_{L^p(J;\mathcal{B})}}{\|\psi_n\|_{L^p(J;\mathcal{B})}} = \lim_{n \rightarrow \infty} \frac{\|\psi_0\|_{L^p(J;\mathcal{B})} M(p)^{n+1} t^{n+1}}{[(n+1)!]^{1/p}} \cdot \frac{(n!)^{1/p}}{\|\psi_0\|_{L^p(J;\mathcal{B})} M(p)^n t^n} \\ &= M(p) t \lim_{n \rightarrow \infty} \frac{1}{(n+1)^{1/p}} = 0 \end{aligned} \quad (95)$$

where, $M(p) = \frac{D}{p^{1/p}}$. Thus $L = 0 < 1$, and by the Ratio Test theorem, the Neumann series converges absolutely. Therefore, the Neumann series converges in the topology of $L^p(I; \mathcal{B})$. \square

5. Applications of the Volterra Theorem

In this section, we will present several different types of applications of theorems 9 and 10. The first example is classical and the second example is the unitary quantum-mechanical example. The closest example to quantum mechanics is example 2 where the spatial operator is a unitary operator. Each example presents two versions, corresponding to the L^∞ Volterra and L^p Volterra theorems, respectively. The following examples are not new and are well-known in the literature. These examples serve as quick applications of the Volterra theory presented in section 5, with serious applications delayed to later papers

Let I be an interval in the temporal dimension such that $I = (0, T)$ for some $T > 0$. In the following examples, the time interval I will always be the same time interval $(0, T)$. Furthermore, let $L^{n,m}(I; \mathbb{R}^d)$ be the Banach space of $L^m(\mathbb{R}^d)$ functions over I . Thus we will define the Lebesgue space $L^{n,m}(I; \mathbb{R}^d)$ as

$$L^{n,m}(I; \mathbb{R}^d) = \left\{ \phi : \left(\int_I \left[\int_{\mathbb{R}^d} |\phi(y, \tau)|^m dy \right]^{n/m} d\tau \right)^{1/n} = \|\phi\|_{L^{n,m}(I; \mathbb{R}^d)} < \infty \right\}. \quad (96)$$

If m and n are equal, then the Lebesgue space $L^{n,m}(I; \mathbb{R}^d)$ will be written as $L^n(I; \mathbb{R}^d)$.

5.1. Example 1

L^∞ Case: Let the Banach space \mathcal{B} be $L^2(\mathbb{R}^n)$ and consider a bounded integrable (e.g., continuous) real or complex-valued kernel $A(t, \tau)$, satisfying the Volterra condition in

(t, τ) . The Hilbert-Schmidt kernel is a function $K : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{F}$ on the space variables, where $\mathbb{F} = \{\mathbb{C}, \mathbb{R}\}$. The norm of the Hilbert-Schmidt kernel is given by

$$\left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |K(x, t; y, \tau)|^2 dx dy \right)^{1/2} = \|K(t, \tau)\|_{L^2(\mathbb{R}^{2n})} \leq N < \infty \quad (97)$$

The linear operator $A(t, \tau)$ is defined on $L^\infty(I^2)$, and $A(t, \tau)$ is a Hilbert-Schmidt operator. Then the Hilbert-Schmidt operator $A(t, \tau) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is given by

$$A(t, \tau)\phi(t) = \int_{\mathbb{R}^n} K(x, t; y, \tau)\phi(y, \tau) dy \quad \forall \phi \in L^{\infty,2}(I; \mathbb{R}^n) \quad (98)$$

It follows that the operator $A(t, \tau)$ is bounded. The function $K(x, t; y, \tau)$ belongs to $L^{\infty,2}(I^2; \mathbb{R}^{2n})$. Then we take the absolute values of $A(t, \tau)\phi(t)$ and we obtain

$$\begin{aligned} |A(t, \tau)\phi(t)| &\leq \int_{\mathbb{R}^n} |K(x, t; y, \tau)| |\phi(y, \tau)| dy \\ &\leq \left(\int_{\mathbb{R}^n} |K(x, t; y, \tau)|^2 dy \right)^{1/2} \left(\int_{\mathbb{R}^n} |\phi(y, \tau)|^2 dy \right)^{1/2} \end{aligned} \quad (99)$$

and hence,

$$\|A(t, \tau)\phi(t)\|_{L^2(\mathbb{R}^n)} \leq \|K(t, \tau)\|_{L^2(\mathbb{R}^{2n})} \|\phi(\tau)\|_{L^2(\mathbb{R}^n)} \leq N \|\phi(\tau)\|_{L^2(\mathbb{R}^n)} \quad (100)$$

with

$$N \equiv \|K\|_{L^{\infty,2}(I^2; \mathbb{R}^{2n})} \quad (101)$$

Therefore by the L^∞ Volterra Theorem, the Volterra integral equation with a Hilbert-Schmidt kernel $K(x, t; y, \tau) \in L^{\infty,2}(I^2; \mathbb{R}^n)$ can be solved by successive approximations.

5.1.1. L^1 Case: Now we will provide an example for the L^1 Volterra Theorem. The difference between this example and the previous one is that the Lebesgue space in time is $L^1(I)$ rather than $L^\infty(I)$. We will assume the same hypotheses for the Volterra kernel $A(t, \tau)$ and the kernel $K(x, t; y, \tau)$ as in the L^∞ case. Thus,

$$\|A(t, \tau)\phi(t)\|_{L^2(\mathbb{R}^n)} \leq D \|\phi(\tau)\|_{L^2(\mathbb{R}^n)} \quad (102)$$

and hence,

$$\|A\phi\|_{L^1(I; \mathbb{R}^n)} \leq \int_0^t D \|\phi(\tau)\|_{L^2(\mathbb{R}^n)} d\tau = D \|\phi\|_{L^{1,2}(I; \mathbb{R}^n)} \quad (103)$$

Thus, we have shown that the norm of $A(t, \tau)\phi(t)$ is bounded, and hence

$$\|\psi\|_{L^1(I; \mathbb{R}^n)} \leq \int_0^t D \|\phi\|_{L^{1,2}(I; \mathbb{R}^n)} d\tau = D \|\phi\|_{L^{1,2}(I; \mathbb{R}^n)} t \quad (104)$$

Therefore by the L^1 Volterra Theorem, the Volterra integral equation with a Hilbert-Schmidt kernel in space and a uniformly bounded kernel in time can be solved by successive approximations.

5.2. Example 2

5.2.1. L^∞ Case: Let $V(x, t)$ be a bounded potential, and $x \in \mathbb{R}^n$. The potential V may be time-dependent, but in that case its bound should be independent of t (i.e., $V \in L^\infty(I; \mathbb{R}^n)$, with $\|V\|_{L^\infty(I; \mathbb{R}^n)} \equiv C$). Let the Banach space \mathcal{B} be the Hilbert space $L^2(\mathbb{R}^n)$. Recall that the solution $u(t)$ is given by the Poisson integral formula

$$u(t) \equiv \int_{\mathbb{R}^n} K_f(x, y, t) h(y) dy \quad (105)$$

where $K_f(x, y, t) = (4\pi it)^{-n/2} e^{i|x-y|^2/4t}$, is the solution of the free Schrödinger equation with initial data $u(x, 0) = h(x)$ in $L^2(\mathbb{R}^n)$.

It is well known that $U_f(t, \tau)$ is unitary, and hence the norm of U_f as an operator from $L^2(\mathbb{R}^n)$ to itself is $\|U_f(t, \tau)\|_{L^2(\mathbb{R}^{2n})} = 1$. A proof that the operator $U_f(t, \tau)$ is a unitary operator can be found in [5, Ch.4]. We wish to solve the Schrödinger equation with the potential V by iteration. The Volterra integral equation is given by

$$u(x, t) + i \int_0^t \hat{U}(t - \tau) V(\tau) u(\tau) d\tau = \hat{U} f(x) \quad (106)$$

where,

$$\hat{U}(t - \tau) V(\tau) u(\tau) = \int_{\mathbb{R}^n} K_f(x, t; y, \tau) V(y, \tau) u(y, \tau) dy \quad (107)$$

Hence, the Volterra theorem applies.

In theorem 9, take $\mathcal{B} = L^2(\mathbb{R}^n)$, $A = UV$ as defined in equation (107). It remains to check that UV is a bounded operator on $L^2(\mathbb{R}^n)$ with bound independent of t and τ . Here $V(\tau)$ is the operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ defined by multiplication of $f(y, \tau)$ by $V(y, \tau)$, and $\|V(\tau)\|$ is its operator norm. But

$$\begin{aligned} \|V(\tau) f(\tau)\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} |V(y, \tau) f(y, \tau)|^2 dy \\ &\leq C^2 \int_{\mathbb{R}^n} |f(y, \tau)|^2 dy = C^2 \|f(\tau)\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (108)$$

Therefore,

$$\|V(\tau) f(\tau)\|_{L^2(\mathbb{R}^n)} \leq C \|f(\tau)\|_{L^2(\mathbb{R}^n)} \quad \forall f \in L^2(\mathbb{R}^n). \quad (109)$$

In other words $\|V\|_{L^\infty(I; \mathbb{R}^n)}$, the norm of the operator $V(\tau) \leq C \equiv \|V\|_{L^\infty(I; \mathbb{R}^n)}$, is the uniform norm of the function $V(x, t)$. Therefore,

$$\|U(t, \tau) V(\tau) f(\tau)\|_{L^2(\mathbb{R}^n)} \leq C \|f(\tau)\|_{L^2(\mathbb{R}^n)}. \quad (110)$$

and the operator norm of $A = UV$ is bounded by $\|U(t, \tau) V(\tau)\|_{L^2(\mathbb{R}^n)} \leq C$. Then,

$$A(t, \tau) f(\tau) = \int_{\mathbb{R}^n} K(x, t; y, \tau) f(y, \tau) dy = \int_{\mathbb{R}^n} K_f(x, t; y, \tau) V(y, \tau) f(y, \tau) dy \quad (111)$$

Therefore, we obtain the following $L^{\infty, 2}$ norm estimate for $\hat{Q}f = SVf$:

$$\|\psi\|_{L^{\infty, 2}(I; \mathbb{R}^n)} = \|SVf\|_{L^{\infty, 2}(I; \mathbb{R}^n)} \leq C \|f\|_{L^{\infty, 2}(I; \mathbb{R}^n)} T \quad (112)$$

where

$$\psi(t) = SVf(t) = \int_0^t U(t, \tau)V(\tau)f(\tau) d\tau \quad (113)$$

Thus we have verified all the hypotheses of the Volterra Theorem, and we conclude that the solution of the Schrödinger equation with potential V is the series $\varphi = \sum_{n=0}^{\infty} \psi_n$, where $\psi_0(t) = f(t) = \hat{U}(t, \tau)h(x)$, and where $h(x)$ is the initial data.

5.2.2. L^1 Case: Now we will consider an application for the L^1 Volterra Theorem. The difference between this example and the previous case is that the Lebesgue space in time is $L^1(I)$ rather than $L^\infty(I)$. We will assume the same hypotheses for the functions V and f as in the L^∞ case. Thus the norm of the potential function V and f is shown to be bounded and the inequality is given by

$$\|V(\tau)f(\tau)\|_{L^2(\mathbb{R}^n)} \leq C\|f(\tau)\|_{L^2(\mathbb{R}^n)} \quad \forall f \in \mathcal{H}. \quad (114)$$

Then, we take the L^1 norm with respect to the time variable and we obtain

$$\|Vf\|_{L^{1,2}(I;\mathbb{R}^n)} \leq C \int_0^t \|f\|_{L^2(\mathbb{R}^n)} d\tau = C\|f\|_{L^{1,2}(I;\mathbb{R}^n)} \quad (115)$$

Also, the operator $A(t, \tau) = U(t, \tau)V(\tau)$ is bounded by $\|U(t, \tau)V(\tau)\|_{L^2(\mathbb{R}^n)} \leq C$. Thus, we have shown that the norm of Vf is bounded, and hence

$$\begin{aligned} \|\psi\|_{L^{1,2}(I;\mathbb{R}^n)} &= \int_0^t \|\psi(\tau)\|_{L^2(\mathbb{R}^n)} d\tau = \int_0^t \left\| \int_0^{t_1} U(t_1, \tau)V(\tau)f(\tau) d\tau \right\|_{L^2(\mathbb{R}^n)} dt_1 \\ &\leq \int_0^t \int_0^{t_1} \|U(t, \tau)V(\tau)\|_{L^2(\mathbb{R}^n)} \|f(\tau)\|_{L^2(\mathbb{R}^n)} d\tau dt_1 \\ &\leq \int_0^t \int_0^{t_1} C\|f(\tau)\|_{L^2(\mathbb{R}^n)} d\tau dt_1 = \int_0^t C\|f\|_{L^{1,2}(I;\mathbb{R}^n)} dt_1 \\ &= C\|f\|_{L^{1,2}(I;\mathbb{R}^n)} t \end{aligned} \quad (116)$$

where $t_1 \leq t$. Therefore by the L^1 Volterra Theorem, the Volterra integral equation with a unitary operator in space can be solved by successive approximations in the topology $L^{1,2}(I; \mathbb{R}^n)$.

6. Conclusion

The similarities between the Schrödinger equation and the heat equation were used to create a theoretical framework which will give the solution to the Schrödinger problem. The L^∞ Volterra theorem proves that Volterra integral equation with a uniformly bounded kernel can be solved by successive approximations with respect to the topology $L^\infty(I; \mathcal{B})$. The L^p Volterra theorem proves the more general case when $L^p(I; \mathcal{B})$, and where $1 < p < \infty$, but it does not prove the special case when $p = 1$. As it turns out, the $p = 1$ case is also proven by inductively applying the L^1 lemma. Thus, the L^1 Volterra theorem proves that the Volterra integral equation with a uniformly bounded kernel can also be solved by successive approximations with respect to the topology $L^1(I; \mathcal{B})$.

In future work I shall apply the Volterra theorem in contexts more complicated than the simple examples presented here. Preliminary work on these applications appears in Chapters 8 and 9 of the thesis [1]. First, I hope to implement an idea due to Balian and Bloch [4] to use a semiclassical propagator to construct a perturbation expansion for a smooth potential $V(x, t)$. The solution of the Schrödinger equation is given in terms of classical paths, and the semiclassical propagator $G_{scl} = Ae^{iS/\hbar}$ to the Green function is considered as the building block for the exact Green function [4]. To prove convergence of the resulting semiclassical Neumann series under suitable technical conditions, in [1, Ch.8] a Semiclassical Volterra Theorem has been proved. There is still more work to be done with regards to applying the Semiclassical Volterra theorem to various types of potentials. An application of the Semiclassical Volterra theorem is the potential problem in \mathbb{R}^n considered in [10].

The Volterra method will also be applied to the boundary value problem for the Schrödinger equation. The double-layer Schrödinger operator will be shown to be bounded from a suitable space of functions defined on the boundary $I \times \partial U$ to itself. At this stage, the boundary value problem remains unresolved.

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